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The model for fracture toughness

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Abstract

It is pointed out that the distribution introduced by Neville [1] (for modeling fracture toughness to failures) is contained by at least two families of distributions known since the 1940s. Some elementary statistical properties of these families are discussed. Six data sets on fracture toughness are used to demonstrate that these families are much better models for fracture toughness than the one introduced by Neville [1].

Keywords: Burr distributions; Fracture toughness

1. Introduction

Neville [1] introduced a new distribution function to characterize fracture toughness statistics. See also Neville [2]. The cumulative probability function (cpf) of this new distribution is given by

$$F(p) = \frac{p^m}{1 + p^m} \tag{1}$$

for $p > 0$ and $m > 0$. This new distribution provided good fits for experimental data on the fracture toughness from a number of materials and different laboratories. The paper also used the distribution to predict the effect of the crack front length, that is of the specimen size.

The distribution given by (1) has received considerable attention in the materials science literature. According to the Science Citation Index, there have been at least 15 papers that have used (1). For the most recent papers, see Papargyris [3], Balankin et al. [4], Lanning and Shen [5] and Smith et al. [6].

In this note, we would like to point out that the distribution given by (1) has been known at least since the 1940s. We discuss two families of distributions that contain (1) as particular cases. We discuss some elementary statistical properties of these two families of distributions. Finally, an application to fracture toughness data is illustrated.

2. Burr type XII distribution

The Burr type XII distribution due to Burr [7] is given by the cpf

$$F(p) = 1 - (1 + p^m)^{-k} \tag{2}$$

for $p > 0$, $m > 0$ and $k > 0$. This is one of the most versatile distributions in statistics. As shown by Rodriguez [8] and Tadikamalla [9], the Burr XII distribution contains the shape characteristics of the normal, log–normal, gamma, logistic and exponential distributions as well as a significant portion of the Pearson type I, II, V, VII, IX and XII families. It has received applications in life testing (see Wingo [10, 11]) and many other areas.

Clearly, (1) is a particular case of (2) for $k = 1$. Other particular cases of (2) include the *F*, inverted beta, Lomax, Pareto and the log–logistic distributions. The probability density function (pdf) corresponding to (2) is:

$$f(p) = kmp^{m-1}(1 + p^m)^{-k-1} \quad .$$

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The nth moment associated with (2) is:

$$E(P^n) = kB\left(k - \frac{n}{m}, 1 + \frac{n}{m}\right),$$

where $B(\cdot, \cdot)$ denotes the beta function defined by

$$B(a,b) = \int_0^1 \omega^{a-1}(1-\omega)^{b-1} dw.$$

In particular, the mean and the variance of P are

$$E(P) = kB\left(k - \frac{1}{m}, 1 + \frac{1}{m}\right)$$

and

$$Var(P) = kB\left(k - \frac{2}{m}, 1 + \frac{2}{m}\right) - k^2 B^2\left(k - \frac{1}{m}, 1 + \frac{1}{m}\right),$$

respectively. If $p_1, p_2, \ldots, p_n$ is a random sample from (2) then the maximum likelihood estimators of the two parameters k and m are the simultaneous solutions of the equations:

$$\sum_{i=1}^{n} \ln(1 + p_i^m) = \frac{n}{k} \quad (3)$$

and

$$(k+1)\sum_{i=1}^{n} \frac{p_i^m \ln p_i}{1 + p_i^m} = \frac{n}{m} + \sum_{i=1}^{n} \ln p_i. \quad (4)$$

For further properties of (2), the reader is referred to Hutchinson and Lai [12] and Johnson et al. [13, 14].

3. Burr type III distribution

The Burr type III distribution also due to Burr [7] is given by the cpf

$$F(p) = (1 + p^{-m})^{-k} \quad (5)$$

for $p > 0$, $m > 0$ and $k > 0$. This distribution is obtained by a simple transformation of (2) and so it retains most of the properties of (2). For instance, it is clear that (1) is a particular case of (5) for $k = 1$. The pdf corresponding to (5) is:

$$f(p) = kmp^{-m-1}(1 + p^{-m})^{-k-1}$$

The nth moment associated with (5) is:

$$E(P^n) = kB\left(k + \frac{n}{m}, 1 - \frac{n}{m}\right).$$

In particular, the mean and the variance of P are

$$E(P) = kB\left(k + \frac{1}{m}, 1 - \frac{1}{m}\right).$$

and

$$Var(P) = kB\left(k + \frac{2}{m}, 1 - \frac{2}{m}\right) - k^2 B^2\left(k + \frac{1}{m}, 1 - \frac{1}{m}\right),$$

respectively. If $p_1, p_2, \ldots, p_n$ is a random sample from (5) then the maximum likelihood estimators of the two parameters k and m are the simultaneous solutions of the equations:

$$\sum_{i=1}^{n} \ln(1 + p_i^{-m}) = \frac{n}{k} \quad (6)$$

and

$$(k+1)\sum_{i=1}^{n} \frac{p_i^{-m} \ln p_i}{1 + p_i^{-m}} = \sum_{i=1}^{n} \ln P_i - \frac{n}{m}. \quad (7)$$

For further properties of (5), the reader is referred to Hutchinson and Lai [12] and Johnson et al. [13, 14].

4. Application

In this section, we demonstrate that the distributions given by (2) and (5) outperform one based on (1). We use fracture toughness data from the six different materials: $Bi_2Sr_2CaCu_2O_{8+x}$, Silicon Nitride (Si_3N_4), Sialon ($Si_{6-x}Al_xO_xN_{8-x}$), Pyroceram 9606, and Titanium Diboride (TiB_2). These data are taken from the web–site:

http://www.ceramics.nist.gov/srd/summary/ftmain.htm.

For the interest of the readers, we have reproduced the data in Table 1.

For each of the six data sets, we fitted Neville's distribution given by (1), Burr type XII distribution given by (2) as well as the Burr type III distribution given by (5). The fitting was performed by the method of maximum likelihood, i.e. solving the equations (3)-(4) and (6)-(7) for fitting (2) and (5), respectively. The parameter estimates as well as the negative values of the maximized log likelihood (NLLH) are given in Table 2.

By the standard likelihood ratio test (see Cox and Hinkley [15]), it is clear that both (2) and (5) outperform Neville's distribution by a long margin. This

Table 1. Fracture toughness data for six different materials.

Material	Fracture toughness data (in the units of MPa $m^{1/2}$)
$Bi_2Sr_3CaCu_2O_{8+x}$	3.2,3.9,2.7,3.2,1.9,1.2,1.8,1.4,1.8,2.9,2.8,2.4
Alumina (Al_2O_3)	5.5,5,4.9,6.4,5.1,5.2,5.2,5,4.7,4,4.5,4.2,4.1,4.56,5.01, 4.7,3.13,3.12,2.68,2.77,2.7,2.36,4.38,5.73,4.35,6.81, 1.91,2.66,2.61,1.68,2.04,2.08,2.13,3.8,3.73,3.71,3.28, 3.9,4,3.8,4.1,3.9,4.05,4,3.95,4,4.5,4.5,4.2,4.55,4.65, 4.1,4.25,4.3,4.5,4.7,5.15,4.3,4.5,4.9,5,5.35,5.15, 5.25,5.8,5.85,5.9,5.75,6.25,6.05,5.9,3.6,4.1,4.5,5.3, 4.85,5.3,5.45,5.1,5.3,5.2,5.3,5.25,4.75,4.5,4.2,4, 4.15,4.25,4.3,3.75,3.95,3.51,4.13,5.4,5,2.1,4.6,3.2, 2.5,4.1,3.5,3.2,3.3,4.6,4.3,4.3,4.5,5.5,4.6,4.9,4.3, 3,3.4,3.7,4.4,4.9,4.9,5
Silicon Nitride (Si_3N_4)	8.3,7.2,3.2,4.96,7.81,6.59,4.9,4.1,4.5,4.7,3.12,2.7,6.75
Sialon ($Si_{6-x}Al_xO_xN_{8-x}$)	3.05,2.9,2.75,2.7,2.65,3.15,3.75,3.8,3.72,3.52,3.44, 3.26,2.99,2.79,3,3.18,3.66,3.2,3.3,3.5,3.1,4.65,3.42, 3.38,3.29
Pyroceram 9606	2.5,3.17,2.69,2.14,2.07,2.8,2.5,2.25
Titanium Diboride (TiB_2)	3.7,5.75,4.25,6.4,4.87,2.3,5.14,4.6,6,5.2,5.36

Table 2. Parameter estimates and NLLH values for the models (1), (2) and (5).

Material	Model (1)		Model (2)			Model (5)		
	$\hat{m}$	NLLH	$\hat{m}$	$\hat{k}$	NLLH	$\hat{m}$	$\hat{k}$	NLLH
$Bi_2Sr_2CaCu_2O_{8+x}$	1.775	26.484	0.015	78.162	19.783	8.368	3.099	15.304
Alumina (Al_2O_3)	1.068	392.779	0.031	22.811	332.126	52.050	3.060	209.768
Silicon Nitride (Si_3N_4)	0.951	46.728	0.046	13.668	40.053	74.383	3.009	26.578
Sialon ($Si_{6-x}Al_xO_xN_{8-x}$)	1.303	71.229	0.056	15.116	58.718	17946.91	8.730	12.658
Pyroceram 9606	1.683	18.610	0.054	20.457	14.577	1563.081	8.676	2.568
Titanium Diboride (TiB_2)	0.987	38.476	0.036	18.130	32.886	62.186	2.941	21.108

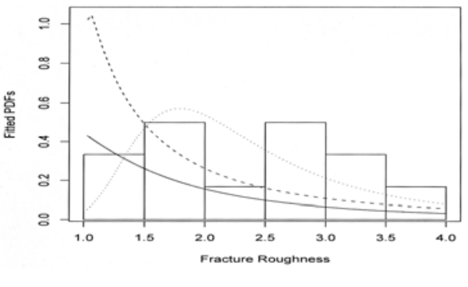

Fig. 1. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on $Bi_2Sr_2CaCu_2O_{8+x}$ are used.

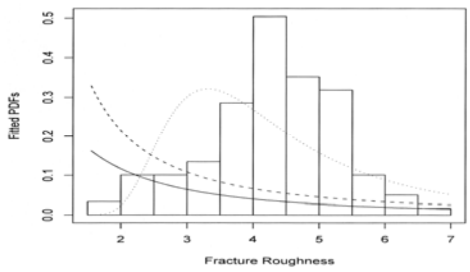

Fig. 2. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on Alumina (Al_2O_3) are used.

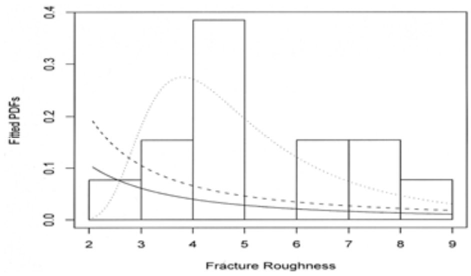

Fig. 3. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on Silicon Nitride (Si_3N_4) are used.

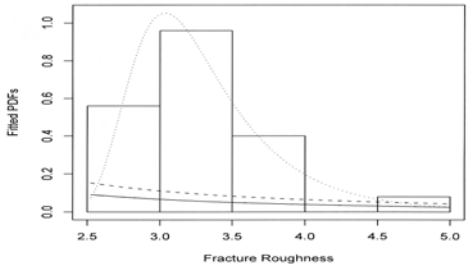

Fig. 4. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on Sialon ($Si_{6-x}Al_xO_xN_{8-x}$) are used.

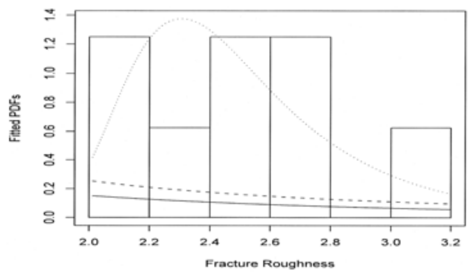

Fig. 5. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on Pyroceram 9606 are used.

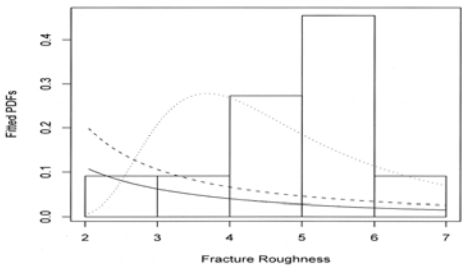

Fig. 6. Fitted pdfs of Neville's distribution (solid curve), Burr type XII distribution (curve of lines) and Burr type III distribution (curve of dots). Fracture toughness data (in the units of Mpa $m^{1/2}$) on Titanium Diboride (TiB_2) are used.

is the case for each of the six data sets. Among (2) and (5), (5) appears to provide the better fit. These findings are confirmed by Figs. 1-6, where the fitted pdfs for all three models are plotted.

5. Conclusions

The purpose of this note is not just to provide priority statements for the model used in the paper by Neville.

In this note, we have introduced two flexible families of distributions that contain Neville's distribution as a particular case. We have discussed some preliminary statistical properties of these two families. We have shown that these families are much better models for fracture toughness data than one based on Neville's distribution.

We feel that the results as well as the references mentioned above can be of assistance in modeling problems of the type considered by Neville (1990).

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